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Regular matrices and their strong preservers over semirings

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Abstract

An $m \times n$ matrix A over a semiring $\mathbb{S}$ is called *regular* if there is an $n \times m$ matrix G over $\mathbb{S}$ such that $AGA = A$. We study the problem of characterizing those linear operators T on the matrices over a semiring such that $T(X)$ is regular if and only if X is. Complete characterizations are obtained for many semirings including the Boolean algebra, the nonnegative reals, the nonnegative integers and the fuzzy scalars.

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1. Introduction

A *semiring* (see [4] or [5]) consists of a set $\mathbb{S}$ and two binary operations on $\mathbb{S}$, addition (+) and multiplication ($\cdot$), such that

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(1) $(\mathbb{S}, +)$ is an Abelian monoid (identity denoted by 0);
(2) $(\mathbb{S}, \cdot)$ is a monoid (identity denoted by 1);
(3) multiplication distributes over addition;
(4) $s \cdot 0 = 0 \cdot s = 0$ for all $s \in \mathbb{S}$; and
(5) $1 \neq 0$.

Usually $\mathbb{S}$ denotes both the semiring and the set. Thus all rings with identity are semirings.

One of the most active and fertile subjects in matrix theory during the past 100 years is the linear preserver problem, which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. Although the linear operators concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings ([2,11] and therein).

Regular matrices play a central role in the theory of matrices, and they have many applications in network and switching theory and information theory [3,5,8]. Recently, a number of authors have studied characterizations of regular matrices over semirings [1,3,5,8,9]. But there are no known results on characterizing those linear operators that (strongly) preserve regular matrices over semirings.

In this paper, we study the linear operators that strongly preserve regular matrices over semirings including the binary Boolean algebra, the nonnegative reals, the nonnegative integers and the fuzzy scalars.

2. Preliminaries

We say that a semiring $\mathbb{S}$ is *commutative* if $(\mathbb{S}, \cdot)$ is Abelian; $\mathbb{S}$ is *antinegative* if 0 is the only element to have an additive inverse. Thus, no ring is antinegative except $\{0\}$.

Algebraic terms such as *unit* and *zero-divisor* are defined for semirings as for rings. The following are some examples of semirings which occur in combinatorics. They are all commutative, antinegative and free of zero-divisors.

Let $\mathbb{B} = \{0, 1\}$, then $(\mathbb{B}, +, \cdot)$ is a semiring (a *Boolean algebra*) if

$$0 + 0 = 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \quad \text{and} \quad 1 + 1 = 1 \cdot 1 = 1.$$

Let $\mathbb{C}$ be any chain with lower bound 0 and upper bound 1, then $(\mathbb{C}, +, \cdot) \equiv (\mathbb{C}, \max, \min)$ is a semiring (a *chain semiring*). In particular, if $\mathbb{F}$ is the real interval [0,1], then $(\mathbb{F}, \max, \min)$ is a semiring, the *fuzzy semiring*. If $\mathbb{P}$ is any subring with identity, of $\mathbb{R}$, the reals (under real addition and multiplication), and $\mathbb{P}_+$ denotes the nonnegative part of $\mathbb{P}$, then $\mathbb{P}_+$ is a semiring. In particular, $\mathbb{Z}_+$ (respectively, $\mathbb{R}_+$), the nonnegative integers (respectively, reals), is a semiring.

Let $\mathscr{M}_{m,n}(\mathbb{S})$ denote the set of all $m \times n$ matrices with entries in a semiring $\mathbb{S}$. If $m = n$, we use the notation $\mathscr{M}_n(\mathbb{S})$ instead of $\mathscr{M}_{n,n}(\mathbb{S})$. Algebraic operations on $\mathscr{M}_{m,n}(\mathbb{S})$ and such notions as *linearity* and *invertibility* are also defined as if the underlying scalars were in a field.

Hereafter, unless otherwise specified, $\mathbb{S}$ will denote an arbitrary semiring that is commutative, antinegative and free of zero-divisors.

The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix all of whose entries are 1, and $O_{m,n}$ is the $m \times n$ zero matrix. We will suppress the subscripts on these matrices when the orders are evident from the context and we write I, J and O, respectively. For any matrix A, A^{t} is denoted by the transpose of A. A zero–one matrix in $\mathscr{M}_{m,n}(\mathbb{S})$ with only one equal to 1 is called a *cell*. If the nonzero entry occurs in the ith row and the jth column, we denote the cell by $E_{i,j}$.

A matrix A in $\mathscr{M}_n(\mathbb{S})$ is said to be *invertible* if there is a matrix B in $\mathscr{M}_n(\mathbb{S})$ such that $AB = BA = I$.

In 1952, Luce [6] showed that a matrix A in $\mathscr{M}_n(\mathbb{B})$ possesses a two-sided inverse if and only if A is an orthogonal matrix in the sense that $AA^{\mathrm{t}} = I$, and that, in this case, A^{t} is a two-sided inverse of A. In 1963, Rutherford [10] showed that if a matrix A in $\mathscr{M}_n(\mathbb{B})$ possesses a one-sided inverse, then the inverse is also a two-sided inverse. Furthermore such an inverse, if it exists, is unique and is A^{t}. Also, it is well known that the $n \times n$ permutation matrices are the only $n \times n$ invertible Boolean matrices.

The notion of generalized inverse of an arbitrary matrix apparently originated in the work of Moore (see [7]). Let A be a matrix in $\mathscr{M}_{m,n}(\mathbb{S})$. Consider a matrix $X \in \mathscr{M}_{n,m}(\mathbb{S})$ in the equation

$$AXA = A. \tag{2.1}$$

If (2.1) has a solution X, then X is called a *generalized inverse* of A. Furthermore A is called *regular* if there is a solution of (2.1).

Clearly, J and O are regular in $\mathscr{M}_{m,n}(\mathbb{S})$ because $JGJ = J$ and $OGO = O$, where G is any cell in $\mathscr{M}_{n,m}(\mathbb{S})$. Thus in general, a solution of (2.1), although it exists, is not necessarily unique. Characterizations of regular matrices over semirings have been obtained by several authors [1,3,5,8,9]. Furthermore Plemmons [8] has obtained an algorithm for computing generalized inverses of Boolean matrices under certain conditions.

The following proposition is an immediate consequence of definitions of regular matrix and invertible matrix.

Proposition 2.1. *Let A be a matrix in $\mathscr{M}_{m,n}(\mathbb{S})$. If $U \in \mathscr{M}_m(\mathbb{S})$ and $V \in \mathscr{M}_n(\mathbb{S})$ are invertible, then the following are equivalent*:

(i) *A is regular in $\mathscr{M}_{m,n}(\mathbb{S})$;*
(ii) *UAV is regular in $\mathscr{M}_{m,n}(\mathbb{S})$;*
(iii) *A^{t} is regular in $\mathscr{M}_{n,m}(\mathbb{S})$.*

Also we can easily show that for a matrix $A \in \mathscr{M}_{m,n}(\mathbb{S})$,

$$A \text{ is regular if and only if } \begin{bmatrix} A & O \\ O & B \end{bmatrix} \text{ is regular} \tag{2.2}$$

for all regular matrices $B \in \mathscr{M}_{p,q}(\mathbb{S})$. In particular, all idempotent matrices in $\mathscr{M}_n(\mathbb{S})$ are regular.

For matrices $A, B \in \mathscr{M}_{m,n}(\mathbb{S})$, we say A *dominates* B (written $B \sqsubseteq A$ or $A \sqsupseteq B$) if $a_{i,j} = 0$ implies $b_{i,j} = 0$ for all i and j. If $A, B \in \mathscr{M}_{m,n}(\mathbb{S})$ and $A \sqsupseteq B$, we define $A \setminus B$ to be the matrix C where $c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} \neq 0, \\ a_{i,j} & \text{otherwise.} \end{cases}$

Define an upper triangular matrix Λ_n in $\mathscr{M}_n(\mathbb{S})$ by

$$\Lambda_n = [\lambda_{i,j}] \equiv \left(\sum_{i \leqslant j}^{n} E_{i,j} \right) \setminus E_{1,n} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 0 \\ & 1 & \cdots & 1 & 1 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix}.$$

Then the following lemma shows that Λ_n is not regular for $n \geqslant 3$.

Lemma 2.2. *Λ_n is regular in $\mathscr{M}_n(\mathbb{S})$ if and only if $n \leqslant 2$.*

Proof. Clearly Λ_n is regular for $n \leqslant 2$ because $\Lambda_n I_n \Lambda_n = \Lambda_n$.

Conversely, assume that Λ_n is regular for some $n \geqslant 3$. Then there is a nonzero $B \in \mathscr{M}_n(\mathbb{S})$ such that $\Lambda_n = \Lambda_n B \Lambda_n$. From $0 = \lambda_{1,n} = \sum_{i=1}^{n-1} \sum_{j=2}^{n} b_{i,j}$, all entries of the 2nd column of B are zero except for $b_{n,2}$. From $0 = \lambda_{2,1} = \sum_{i=2}^{n} b_{i,1}$, all entries of the 1st column of B are zero except for $b_{1,1}$. Also, from $0 = \lambda_{3,2} = \sum_{i=3}^{n} \sum_{j=1}^{2} b_{i,j}$, we have $b_{n,2} = 0$. If we combine these three results, we conclude that all entries of the first two columns are zero except for $b_{1,1}$. But then $1 = \lambda_{2,2} = \sum_{i=2}^{n} \sum_{j=1}^{2} b_{i,j} = 0$, a contradiction. Hence Λ_n is not regular for all $n \geqslant 3$. □

In particular, $\Lambda_3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is not regular. Let

$$\Phi_{m,n} = \begin{bmatrix} \Lambda_3 & O \\ O & O \end{bmatrix} \tag{2.3}$$

for all $\min\{m, n\} \geqslant 3$. Then $\Phi_{m,n}$ is not regular by (2.2).

Proposition 2.3. *Let* $\min\{m, n\} \geqslant 3$. *For every cell E in $\mathscr{M}_{m,n}(\mathbb{S})$, there is a regular matrix A such that $E + A$ is not regular.*

Proof. Consider the matrix $\Phi_{m,n}$ in (2.3). Let P and Q be permutation matrices such that $PEQ = E_{1,1}$. Consider a matrix A satisfying $PAQ = E_{1,2} + E_{2,2} + E_{2,3} + E_{3,3}$. Then

$$(PAQ)(G_{2,1} + G_{3,3})(PAQ) = PAQ \quad \text{and} \quad P(E + A)Q = \Phi_{m,n},$$

where $G_{i,j}$ are cells in $\mathscr{M}_{n,m}(\mathbb{S})$. Thus $E + A$ is not regular, while A is regular by Proposition 2.1. □

The *pattern*, A^*, of a matrix $A \in \mathscr{M}_{m,n}(\mathbb{S})$ is the matrix in $\mathscr{M}_{m,n}(\mathbb{B})$ whose (i, j)th entry is 0 if and only if $a_{i,j} = 0$. By the definition, we have

$$(AB)^* = A^* B^* \quad \text{and} \quad (B + C)^* = B^* + C^*$$

for all $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and for all $B, C \in \mathscr{M}_{n,q}(\mathbb{S})$. It follows that if A is regular in $\mathscr{M}_{m,n}(\mathbb{S})$, then A^* is regular in $\mathscr{M}_{m,n}(\mathbb{B})$. Let $\mathscr{R}(\mathbb{S})$ be the set of all regular matrices in $\mathscr{M}_{m,n}(\mathbb{S})$; that is, $\mathscr{R}(\mathbb{S}) = \{X \in \mathscr{M}_{m,n}(\mathbb{S}) | X \text{ is regular}\}$ and let $\mathscr{R}(\mathbb{S})^* = \{Y \in \mathscr{M}_{m,n}(\mathbb{B}) | Y = X^* \text{ for some } X \in \mathscr{R}(\mathbb{S})\}$. In general, $\mathscr{R}(\mathbb{S})^* \neq \mathscr{R}(\mathbb{B})$. For example, consider the matrix $Y = E_{1,1} + E_{1,2} + E_{2,2}$. Then we can easily check that $Y \in \mathscr{R}(\mathbb{B})$ but $Y \notin \mathscr{R}(\mathbb{R}_+)^*$.

The number of nonzero entries of a matrix A is denoted by $\sharp(A)$.

Proposition 2.4. *Let A be a matrix in $\mathscr{M}_{m,n}(\mathbb{S})$ with $\sharp(A) = 5$ such that A has a row or a column that has at least three nonzero entries. If $E_{1,1} + E_{1,2} + E_{2,2} \in \mathscr{R}(\mathbb{S})^*$, then $A^* \in \mathscr{R}(\mathbb{S})^*$.*

Proof. By hypothesis, $E_{1,1} + E_{1,2} + E_{2,2} \in \mathscr{R}(\mathbb{S})^*$, thus there exist $a, b, c, x, y, z, w \in \mathbb{S}$ such that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \oplus O$ is a regular matrix with a generalized inverse $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \oplus O$ (in fact, $z = 0$) where O is the zero matrix of appropriate order. Fix a, b, c, x, y, z, w.

Assume that A has a row that has at least three nonzero entries. By Proposition 2.1, we may permute the rows and columns and/or transpose a regular matrix and the result is a regular matrix. Thus there are 11 inequivalent matrices in $\mathscr{M}_{m,n}(\mathbb{B})$ to consider:

$$X_1 = E_{1,1} + E_{1,2} + E_{1,3} + E_{1,4} + E_{1,5}, \quad X_2 = E_{1,1} + E_{1,2} + E_{1,3} + E_{1,4} + E_{2,5},$$
$$X_3 = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,4} + E_{2,5}, \quad X_4 = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,3} + E_{2,4},$$
$$X_5 = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,4} + E_{3,5}, \quad X_6 = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,4} + E_{3,4},$$
$$X_7 = E_{1,1} + E_{1,2} + E_{1,3} + E_{1,4} + E_{2,4}, \quad X_8 = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,2} + E_{2,3},$$
$$X_9 = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,3} + E_{3,4}, \quad X_{10} = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,3} + E_{3,3},$$
$$X_{11} = E_{1,1} + E_{1,2} + E_{1,3} + E_{2,2} + E_{3,3}.$$

For $i = 1, \dots, 6$ let A_i be the (0, 1)-matrix in $\mathscr{M}_{m,n}(\mathbb{S})$ whose pattern is X_i.

For $i = 7, \dots, 11$, let

$$A_7 = a(E_{1,1} + E_{1,2} + E_{1,3}) + bE_{1,4} + cE_{2,4}, \quad A_8 = aE_{1,1} + b(E_{1,2} + E_{1,3}) + c(E_{2,2} + E_{2,3}),$$
$$A_9 = a(E_{1,1} + E_{1,2}) + bE_{1,3} + cE_{2,3} + E_{3,4}, \quad A_{10} = a(E_{1,1} + E_{1,2}) + bE_{1,3} + c(E_{2,3} + E_{3,3}),$$
$$A_{11} = aE_{1,1} + b(E_{1,2} + E_{1,3}) + c(E_{2,2} + E_{3,3}).$$

So that $A_i^* = X_i$, $i = 1, \dots, 11$.

Now let

$$B_1 = G_{1,1}, \quad B_2 = G_{1,1} + G_{5,2},$$
$$B_3 = G_{1,1} + G_{4,2}, \quad B_4 = G_{1,1} + G_{4,2},$$
$$B_5 = G_{1,1} + G_{4,2} + G_{5,3}, \quad B_6 = G_{1,1} + G_{4,2},$$
$$B_7 = xG_{3,1} + yG_{3,2} + wG_{4,2} \quad B_8 = xG_{1,1} + yG_{1,2} + wG_{2,2},$$
$$B_9 = xG_{2,1} + yG_{2,2} + wG_{3,2} + G_{4,3} \quad B_{10} = xG_{2,1} + yG_{2,2} + wG_{3,2},$$
$$B_{11} = xG_{1,1} + y(G_{1,2} + G_{1,3}) + w(G_{2,2} + G_{3,3}),$$

where $G_{i,j}$ are cells in $\mathscr{M}_{n,m}(\mathbb{S})$.

Routine calculations show that $A_iB_iA_i = A_i$ for $i = 1, \dots, 11$ so that $A_i \in \mathscr{R}(\mathbb{S})$ and hence $A_i^* \in \mathscr{R}(\mathbb{S})^*$.

This completes the proof. □

The (*factor*) *rank*, $\mathrm{fr}(A)$, of a nonzero $A \in \mathscr{M}_{m,n}(\mathbb{S})$ is defined as the least integer r for which there are $B \in \mathscr{M}_{m,r}(\mathbb{S})$ and $C \in \mathscr{M}_{r,n}(\mathbb{S})$ such that $A = BC$, see [2,4]. The rank of a zero matrix is zero. Also we can easily obtain

$$0 \leqslant \mathrm{fr}(A) \leqslant \min\{m, n\} \quad \text{and} \quad \mathrm{fr}(AB) \leqslant \min\{\mathrm{fr}(A), \mathrm{fr}(B)\} \tag{2.4}$$

for all $A \in \mathscr{M}_{m,n}(\mathbb{S})$ and for all $B \in \mathscr{M}_{n,q}(\mathbb{S})$.

Proposition 2.5. *Let* $\min\{m, n\} \geqslant 3$. *If A is a matrix in* $\mathscr{M}_{m,n}(\mathbb{S})$ *with* $\sharp(A) = 3$ *and* $\mathrm{fr}(A) = 2$ *or* 3, *then there is a matrix B with* $\sharp(B) = 2$ *such that* $(A + B)^* \notin \mathscr{R}(\mathbb{S})^*$.

Proof. Since $\sharp(A) = 3$ and $\mathrm{fr}(A) = 2$ or 3, there are permutations P and Q such that $PAQ \sqsubseteq \Phi_{m,n}$. Let C be a matrix in $\mathscr{M}_{m,n}(\mathbb{S})$ with $\sharp(C) = 2$ such that $(PAQ + C)^* = \Phi_{m,n}$. If we take $B = P^T CQ^T$, then $(A + B)^* = P^T \Phi_{m,n} Q^T \notin \mathscr{R}(\mathbb{B})$ by Proposition 2.1 and hence $(A + B)^* \notin \mathscr{R}(\mathbb{S})^*$. □

Linearity of operators on $\mathscr{M}_{m,n}(\mathbb{S})$ is defined as for vector spaces over fields. A linear operator on $\mathscr{M}_{m,n}(\mathbb{S})$ is completely determined by its behavior on the set of cells in $\mathscr{M}_{m,n}(\mathbb{S})$.

If T is a linear operator on $\mathscr{M}_{m,n}(\mathbb{S})$, let T^*, its *pattern*, be the linear operator on $\mathscr{M}_{m,n}(\mathbb{B})$ defined by $T^*(E_{i,j}) = [T(E_{i,j})]^*$ for all cells $E_{i,j}$. Since $\mathbb{S}$ is a semiring that is commutative, antinegative and free of zero-divisors, we have $T^*(A) = [T(A)]^*$ for all $A \in \mathscr{M}_{m,n}(\mathbb{S})$.

Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{S})$. We say that

(1) T *preserves regularity* (or T *preserves* $\mathscr{R}(\mathbb{S})$) if $T(A) \in \mathscr{R}(\mathbb{S})$ whenever $A \in \mathscr{R}(\mathbb{S})$;
(2) T *strongly preserves regularity* (or T *strongly preserves* $\mathscr{R}(\mathbb{S})$) when $T(A) \in \mathscr{R}(\mathbb{S})$ if and only if $A \in \mathscr{R}(\mathbb{S})$ for all $A \in \mathscr{M}_{m,n}(\mathbb{S})$;
(3) T is *singular* if $T(X) = O$ for some nonzero X; Otherwise T is *nonsingular*.

Example 2.6. Let A be any nonzero regular matrix in $\mathscr{M}_{m,n}(\mathbb{S})$, where $\mathbb{S} = \mathbb{B}$ or $\mathbb{C}$. Define a linear operator T on $\mathscr{M}_{m,n}(\mathbb{S})$ by

$$T(X) = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} x_{i,j}\right) A$$

for all $X \in \mathscr{M}_{m,n}(\mathbb{S})$. Then we can easily show that T is a nonsingular linear operator that preserves regularity. But T does not preserve nonregular matrices. Hence T does not strongly preserve regularity. □

Lemma 2.7. *Let* $\min\{m, n\} \geqslant 3$. *If T strongly preserves regularity on $\mathscr{M}_{m,n}(\mathbb{S})$, then T is nonsingular.*

Proof. If $T(X) = O$ for some nonzero $X \in \mathscr{M}_{m,n}(\mathbb{S})$, then $T(E) = O$ for all cells $E \sqsubseteq X$. For such E, there is a matrix A such that $A \in \mathscr{R}(\mathbb{S})$ and $E + A \notin \mathscr{R}(\mathbb{S})$ by Proposition 2.3. Nevertheless, $T(E + A) = T(A)$, a contradiction to the fact that T strongly preserves regularity. Hence $T(X) \neq O$ for all nonzero X. Therefore T is nonsingular. □

Let A and B be matrices in $\mathscr{M}_{m,n}(\mathbb{S})$. Then the matrix $A \circ B$ denotes the *Hadamard product* (or *Schur product*). That is, the (i, j)th entry of $A \circ B$ is $a_{i,j}b_{i,j}$.

Lemma 2.8. *Let* $\min\{m, n\} \geqslant 3$ *and* $B \in \mathscr{M}_{m,n}(\mathbb{S})$. *Suppose that T is a linear operator on $\mathscr{M}_{m,n}(\mathbb{S})$ defined by $T(X) = X \circ B$ for all $X \in \mathscr{M}_{m,n}(\mathbb{S})$. If T strongly preserves regularity, then all entries of B are nonzero and regular. In particular if* $\mathrm{fr}(B) = 1$, *then there are diagonal matrices D and E such that $T(X) = DXE$ for all $X \in \mathscr{M}_{m,n}(\mathbb{S})$.*

Proof. By Lemma 2.7, all $b_{i,j}$ are nonzero. Let $b_{i,j}$ be any entry in B. Then $E_{i,j} \circ B = b_{i,j}E_{i,j}$ is regular because $E_{i,j}$ is. Thus there is a matrix $A = [a_{k,l}] \in \mathscr{M}_{n,m}(\mathbb{S})$ such that $(E_{i,j} \circ B)A(E_{i,j} \circ B) = E_{i,j} \circ B$ so that $b_{i,j}a_{j,i}b_{i,j} = b_{i,j}$. Since $b_{i,j}$ is arbitrary, all entries of B are regular.

If $\mathrm{fr}(B) = 1$, then there are matrices $M = [d_1, \dots, d_m]^{\mathrm{t}} \in \mathscr{M}_{m,1}(\mathbb{S})$ and $N = [e_1, \dots, e_n] \in \mathscr{M}_{1,n}(\mathbb{S})$ such that $B = MN$. Let $D = \mathrm{diag}(d_1, \dots, d_m)$ and $E = \mathrm{diag}(e_1, \dots, e_n)$. Then the (i, j)th entry of $T(X)$ is $b_{i,j}x_{i,j}$ and the (i, j)th entry of DXE is $d_i x_{i,j} e_j = b_{i,j}x_{i,j}$. Thus the result follows. □

3. Main results

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ be a matrix in $\mathscr{M}_{m,n}(\mathbb{B})$, where $\mathbf{a}_j$ denotes the jth column of A for all $j = 1, \dots, n$. Then the *column space* of A is the set $\left\{\sum_{j=1}^{n} \alpha_j \mathbf{a}_j | \alpha_j \in \mathbb{B}\right\}$, and denoted by $\langle A\rangle$; the *row space* of A is $\langle A^{\mathrm{t}}\rangle$.

For $A \in \mathscr{M}_{m,n}(\mathbb{B})$ with $\mathrm{fr}(A) = k$, A is said to be *space decomposable* if there are matrices $B \in \mathscr{M}_{m,k}(\mathbb{B})$ and $C \in \mathscr{M}_{k,n}(\mathbb{B})$ such that $A = BC$, $\langle A\rangle = \langle B\rangle$ and $\langle A^{\mathrm{t}}\rangle = \langle C^{\mathrm{t}}\rangle$.

Theorem 3.1 [9]. *Let A be a matrix in $\mathscr{M}_{m,n}(\mathbb{B})$. Then A is regular if and only if A is space decomposable.*

Lemma 3.2. *If A is a matrix in $\mathscr{M}_{m,n}(\mathbb{B})$ with $\mathrm{fr}(A) \leqslant 2$, then A is regular.*

Proof. If $\mathrm{fr}(A) = 0$, $A = O$ is clearly regular. If $\mathrm{fr}(A) = 1$, there are permutation matrices P and Q such that $PAQ = \begin{bmatrix} J & O \\ O & O \end{bmatrix}$. By (2.2) and Proposition 2.1, A is regular.

If $\mathrm{fr}(A) = 2$, there are matrices $B = [\mathbf{b}_1\ \mathbf{b}_2]$ and $C = [\mathbf{c}_1\ \mathbf{c}_2]^{\mathrm{t}}$ of orders $m \times 2$ and $2 \times n$, respectively such that $A = BC$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are distinct nonzero columns of B, and $\mathbf{c}_1$ and $\mathbf{c}_2$ are distinct nonzero columns of C^{t}. Then all columns of A are of the forms $\mathbf{0}$, $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_1 + \mathbf{b}_2$ so that $\langle A\rangle = \langle B\rangle$. Similarly, $\langle A^{\mathrm{t}}\rangle = \langle C^{\mathrm{t}}\rangle$. Therefore A is space decomposable and hence A is regular by Theorem 3.1. $\square$

Theorem 3.3. *Let $\min\{m, n\} \leqslant 2$. If T is an operator (that need not be linear) on $\mathscr{M}_{m,n}(\mathbb{B})$, then T strongly preserves regularity.*

Proof. If $\min\{m, n\} \leqslant 2$, then all matrices in $\mathscr{M}_{m,n}(\mathbb{B})$ are regular by Lemma 3.2 and (2.4). Hence $T(A)$ is always regular for all $A \in \mathscr{M}_{m,n}(\mathbb{B})$. Thus the result follows. $\square$

Proposition 3.4. *Let $A \in \mathscr{M}_{m,n}(\mathbb{S})$ be a sum of k cells with $\mathrm{fr}(A) = k$, where $\min\{m, n\} \geqslant 3$ and $2 \leqslant k \leqslant \min\{m, n\}$. Then $J \setminus A \in \mathscr{R}(\mathbb{S})^*$ if and only if $k = 2$. In particular, $J \setminus A \notin \mathscr{R}(\mathbb{S})$ for $k \geqslant 3$.*

Proof. Without loss of generality, we assume that $\min\{m, n\} = m \geqslant 3$ and $A = \sum_{t=1}^{k} E_{t,t}$. If $k = 2$, consider a matrix $X = \begin{bmatrix} X' & J_{2,n-2} \\ J_{m-2,2} & 2J_{m-2,n-2} \end{bmatrix} \in \mathscr{M}_{m,n}(\mathbb{S})$, where $X' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $X(G_{1,2} + G_{2,1})X = X$, where $G_{i,j}$ are cells in $\mathscr{M}_{n,m}(\mathbb{S})$; that is, $X \in \mathscr{R}(\mathbb{S})$. Hence $J \setminus A(= X^*) \in \mathscr{R}(\mathbb{S})^*$.

Let $k \geqslant 3$. Now we will show that $Y = J \setminus A \notin \mathscr{R}(\mathbb{B})$. If not, there is a nonzero $B \in \mathscr{M}_{n,m}(\mathbb{B})$ such that $Y = YBY$. Then the (t, t)th entry of YBY becomes

$$\sum_{i\in I}\sum_{j\in J} b_{i,j} \tag{3.1}$$

for all $t = 1, \dots, k$, where $I = \{1, \dots, n\} \setminus \{t\}$ and $J = \{1, \dots, m\} \setminus \{t\}$. From $y_{1,1} = 0$ and (3.1), we have

$$b_{i,j} = 0 \quad \text{for all } i = 2, \dots, n;\quad j = 2, \dots, m. \tag{3.2}$$

Consider the 1st row and the 1st column of B. It follows from $y_{2,2} = 0$ and (3.1) that

$$b_{i,1} = 0 = b_{1,j} \quad \text{for all } i = 1, 3, 4, \dots, n;\quad j = 1, 3, 4, \dots, m. \tag{3.3}$$

Also, from $y_{3,3} = 0$, we obtain $b_{1,2} = b_{2,1} = 0$, and hence $B = O$ by (3.2) and (3.3), a contradiction. Thus $J \setminus A \notin \mathscr{R}(\mathbb{B})$, equivalently $Z \notin \mathscr{R}(\mathbb{S})$ for all $Z \in \mathscr{M}_{m,n}(\mathbb{S})$ with $Z^* = J \setminus A$. Hence $J \setminus A \notin \mathscr{R}(\mathbb{S})^*$. $\square$

As shown in Example 2.6, if $\min\{m, n\} \geqslant 3$, there is a linear operator on $\mathscr{M}_{m,n}(\mathbb{B})$ such that T preserves regularity, while T does not strongly preserve regularity.

The next propositions, lemmas and their corollaries are necessary to prove the main theorem. In the following, we assume that $\min\{m, n\} = m \geqslant 3$ and T is a linear operator on $\mathscr{M}_{m,n}(\mathbb{S})$ that strongly preserves $\mathscr{R}(\mathbb{S})$. Then we can easily show that T^* strongly preserves $\mathscr{R}(\mathbb{S})^*$.

Let

$$k_{\max} = \max\{\sharp(N) : N \in \mathscr{M}_{m,n}(\mathbb{B}) \quad \text{and} \quad N \notin \mathscr{R}(\mathbb{S})^*\}$$

and

$$\mathscr{N} = \{N \in \mathscr{M}_{m,n}(\mathbb{B}) : N \notin \mathscr{R}(\mathbb{S})^* \quad \text{with} \quad \sharp(N) = k_{\max}\}.$$

Since $J \in \mathscr{R}(\mathbb{S})$, $k_{\max} < mn$, and so by Proposition 3.4, $mn - 3 \leqslant k_{\max} \leqslant mn - 1$.

Proposition 3.5. *For distinct cells, E and F, $T^*(E) \not\sqsubseteq T^*(F)$. In particular, if $\sharp(T^*(E)) = \sharp(T^*(F)) = 1$, then $T^*(E) \neq T^*(F)$.*

Proof. Suppose $T^*(E) \sqsubseteq T^*(F)$ for some distinct cells E and F. Then there are cells E_1 and E_2 different from F such that $\operatorname{fr}(E + E_1 + E_2) = 3$. Since $F \sqsubseteq J \setminus (E + E_1 + E_2)$, we have $T^*(E) \sqsubseteq T^*(F) \sqsubseteq T^*(J \setminus (E + E_1 + E_2))$ so that

$$T^*(J \setminus (E_1 + E_2)) = T^*(E) + T^*(J \setminus (E + E_1 + E_2)) = T^*(J \setminus (E + E_1 + E_2)).$$

But this is impossible as $J \setminus (E + E_1 + E_2) \notin \mathscr{R}(\mathbb{S})^*$ while $J \setminus (E_1 + E_2) \in \mathscr{R}(\mathbb{S})^*$ by Proposition 3.4. $\square$

In the following four lemmas, we will prove

$$T^*(\mathscr{N}) \subseteq \mathscr{N}.$$

Lemma 3.6. *If $k_{\max} = mn - 1$, then $T^*(\mathscr{N}) \subseteq \mathscr{N}$.*

Proof. If $k_{\max} = mn - 1$, then $\mathscr{N} = \{J \setminus E \in \mathscr{M}_{m,n}(\mathbb{B}) : E \text{ is a cell}\}$. It suffices to show $\sharp(T^*(J \setminus E)) = mn - 1$ for all cells E. If $\sharp(T^*(J \setminus E)) = mn$ for some cell E, then $T^*(J \setminus E) = J \in \mathscr{R}(\mathbb{S})^*$ which contradicts $J \setminus E \notin \mathscr{R}(\mathbb{S})^*$. Next suppose $\sharp(T^*(J \setminus E)) < mn - 1$ for some cell E. Then there is a matrix C with $C \sqsubseteq J \setminus E$ and $\sharp(C) < mn - 1$ such that $T^*(C) = T^*(J \setminus E)$. Take a cell F different from E such that $F \not\sqsubseteq C$. It follows from $(J \setminus E) + (J \setminus (C + F)) = J$ that

$$T^*(J) = T^*(J \setminus E) + T^*(J \setminus (C + F)) = T^*(C) + T^*(J \setminus (C + F)) = T^*(J \setminus F).$$

But this is impossible as $J \setminus F \notin \mathscr{R}(\mathbb{S})^*$ while $J \in \mathscr{R}(\mathbb{S})^*$. Thus the result follows. $\square$

Lemma 3.7. *If $m = n = 3$ and $k_{\max} = 7$, then $T^*(\mathscr{N}) \subseteq \mathscr{N}$.*

Proof. In this case, $\mathscr{N} = \{N \in \mathscr{M}_3(\mathbb{B}) : N \notin \mathscr{R}(\mathbb{S})^* \text{ and } \sharp(N) = 7\}$. Let $N \in \mathscr{N}$ be arbitrary. It suffices to show $\sharp(T^*(N)) = 7$. It follows from $k_{\max} = 7$ that $\sharp(T^*(N)) \leqslant 7$. Suppose $\sharp(T^*(N)) \leqslant 6$. Write $N = \sum_{i=1}^{7} E_i$ and $J = \sum_{i=1}^{9} E_i$ for cells $E_1, \ldots, E_9$. By Lemma 2.7, $\sharp(T^*(E_i)) \geqslant 1$ for all i. If $\sharp(T^*(E_i)) = 1$ for all $i = 1, \ldots, 7$, then by Proposition 3.5, $T^*(E_1), \ldots, T^*(E_7)$ are distinct cells. But then

$$7 = \sharp(T^*(E_1) + \cdots + T^*(E_7)) = \sharp(T^*(E_1 + \cdots + E_7)) = \sharp(T^*(N)) \leqslant 6,$$

a contradiction. Hence one of $T^*(E_i)$, say $T^*(E_1)$, has at least two nonzero entries. Since $\sharp(T^*(N)) \leqslant 6$, we can find four cells in $\{E_2, \ldots, E_7\}$, say $E_2, \ldots, E_5$ such that $T^*(E_1 + \cdots + E_5) = T^*(N)$.

Notice that by Proposition 3.4, $\mathrm{fr}(J \setminus N) = 1$. By Proposition 2.1, we have $X \in \mathscr{N}$ for all $X \in \mathscr{M}_3(\mathbb{B})$ with $\sharp(X) = 7$ and $\mathrm{fr}(J \setminus X) = 1$. If $\mathrm{fr}(E_6 + E_7) = 1$, then

$$\begin{aligned} T^*(J) &= T^*(N) + T^*(E_8 + E_9) = T^*(E_1 + \cdots + E_5) + T^*(E_8 + E_9) \\ &= T^*(J \setminus (E_6 + E_7)), \end{aligned}$$

which is impossible as $J \setminus (E_6 + E_7) \notin \mathscr{R}(\mathbb{S})^*$ while $J \in \mathscr{R}(\mathbb{S})^*$. Thus, $\mathrm{fr}(E_6 + E_7) = 2$. Since $m = n = 3$, there is a cell in $\{E_6, E_7\}$, say E_6, and a cell in $\{E_8, E_9\}$, say E_8 such that $\mathrm{fr}(E_6 + E_8) = 1$. Since $T^*(N) = T^*(E_1 + \cdots + E_5) \sqsubseteq T^*(E_1 + \cdots + E_5 + E_7) \sqsubseteq T^*(N)$, we have $T^*(N) = T^*(E_1 + \cdots + E_5 + E_7)$. But then

$$\begin{aligned} T^*(J \setminus E_8) &= T^*(N) + T^*(E_9) = T^*(E_1 + \cdots + E_5 + E_7) + T^*(E_9) \\ &= T^*(J \setminus (E_6 + E_8)), \end{aligned}$$

which is impossible because $J \setminus E_8 \in \mathscr{R}(\mathbb{S})^*$ while $J \setminus (E_6 + E_8) \notin \mathscr{R}(\mathbb{S})^*$. Thus, $\sharp(T^*(N)) = 7$. $\square$

Lemma 3.8. *If $m = n = 3$ and $k_{\max} = 6$, then $T^*(\mathscr{N}) \subseteq \mathscr{N}$.*

Proof. In this case, $\mathscr{N} = \{N \in \mathscr{M}_3(\mathbb{B}) : N \notin \mathscr{R}(\mathbb{S})^* \text{ and } \sharp(N) = 6\}$. Let $N \in \mathscr{N}$ be arbitrary. It suffices to show $\sharp(T^*(N)) = 6$. It follows from $k_{\max} = 6$ that $\sharp(T^*(N)) \leqslant 6$. Suppose $\sharp(T^*(N)) \leqslant 5$. Write $N = \sum_{i=1}^{6} E_i$ and $J = \sum_{i=1}^{9} E_i$ for cells $E_1, \ldots, E_9$. By Lemma 2.7, $\sharp(T^*(E_i)) \geqslant 1$ for all i. We will claim that there are distinct cells E_i, E_j, E_k in $\{E_1, \ldots, E_6\}$ such that

$$T^*(E_i) + T^*(E_j) + T^*(E_k) = T^*(E_i + E_j + E_k) = T^*(N).$$

Clearly the claim holds if there is a cell E_i in $\{E_1, \ldots, E_6\}$ such that $\sharp(T^*(E_i)) \geqslant 3$. Suppose $\sharp(T^*(E_i)) \leqslant 2$ for $i = 1, \ldots, 6$. Let $F_i = T^*(E_i)$ for $i = 1, \ldots, 6$. Without loss of generality, we may assume that

$$\sharp(F_1) = \cdots = \sharp(F_r) = 2 \quad \text{and} \quad \sharp(F_{r+1}) = \cdots = \sharp(F_6) = 1$$

for some r. If $r = 0$ or $r = 1$, then by Proposition 3.5, we see that $F_1, \ldots, F_6$ are all disjoint and hence

$$\sharp(T^*(N)) = \sharp(F_1 + \cdots + F_6) = \sharp(F_1) + \cdots + \sharp(F_6) \geqslant 6,$$

which is impossible. Thus, $r \geqslant 2$. Now suppose $\sharp(F_i + F_j) \leqslant 3$ for all $1 \leqslant i < j \leqslant r$. Then there is a cell G such that $G \sqsubseteq F_i$ for all $i = 1, \ldots, r$. It follows from Proposition 3.5 that the six cells $F_1 \setminus G, \ldots, F_r \setminus G, F_{r+1}, \ldots, F_6$ are distinct and so

$$\begin{aligned} \sharp(T^*(N)) &= \sharp(F_1 + \cdots + F_6) \geqslant \sharp(F_1 \setminus G) + \cdots + \sharp(F_r \setminus G) + \sharp(F_{r+1}) + \cdots + \sharp(F_6) \\ &= 6, \end{aligned}$$

which is impossible. Thus there are two cells in $\{F_1, \ldots, F_r\}$, say F_1 and F_2, such that $\sharp(F_1 + F_2) = 4$. In this case, we can always find another cell F_k in $\{F_3, \ldots, F_6\}$ such that $F_1 + F_2 + F_k = T^*(N)$. So our claim holds.

Without loss of generality, we may assume $T^*(E_1 + E_2 + E_3) = T^*(N)$. Since $N \notin \mathscr{R}(\mathbb{S})^*$, we must have $E_1 + E_2 + E_3 \notin \mathscr{R}(\mathbb{S})^*$ and it can be easily checked that this is possible only if

$E_1 + E_2 + E_3 = P(E_{11} + E_{12} + E_{21})Q$ for some permutation matrices P and Q. If $PE_{22}Q \in \{E_4, E_5, E_6\}$, then

$$T^*(N) = T^*(P(E_{11} + E_{12} + E_{21})Q) \sqsubseteq T^*(P(E_{11} + E_{12} + E_{21} + E_{22})Q) \sqsubseteq T^*(N),$$

and so $T^*(P(E_{11} + E_{12} + E_{21} + E_{22})Q) = T^*(N)$ which is impossible as $P(E_{11} + E_{12} + E_{21} + E_{22})Q \in \mathscr{R}(\mathbb{S})^*$. Thus, $PE_{22}Q \notin \{E_4, E_5, E_6\}$. Similarly, we can check $PE_{23}Q, PE_{32}Q \notin \{E_4, E_5, E_6\}$. Therefore, $\{E_4, E_5, E_6\} = \{PE_{13}Q, PE_{31}Q, PE_{33}Q\}$. In particular, we have $T^*(P(E_{11} + E_{12} + E_{21} + E_{13})Q) = T^*(P(E_{11} + E_{12} + E_{21})Q)$. But then

$$\begin{aligned} &T^*(P(J \setminus (E_{22} + E_{31}))Q) \\ &\quad = T^*(P(E_{11} + E_{12} + E_{21} + E_{13})Q) + T^*(P(E_{23} + E_{32} + E_{33})Q) \\ &\quad = T^*(P(E_{11} + E_{12} + E_{21})Q) + T^*(P(E_{23} + E_{32} + E_{33})Q) \\ &\quad = T^*(P(J \setminus (E_{13} + E_{22} + E_{31}))Q), \end{aligned}$$

which is impossible by Proposition 3.4. Therefore $\sharp(T^*(N)) = 6$ and the result follows. □

Lemma 3.9. *Suppose $n \geqslant 4$ and $k_{\max} \leqslant mn - 2$. For any matrix A with $\sharp(A) \leqslant mn - 2$,*

$$\sharp(T^*(A)) \geqslant \sharp(A).$$

Consequently, we have $T^*(\mathscr{N}) \subseteq \mathscr{N}$.

Proof. Let $\sharp(A) = p$. We will prove the first part of the result by induction on p. By Lemma 2.7, the result holds for $p = 1$. Assume that the result holds for all B with $\sharp(B) < p$. Let A be an arbitrary matrix with $\sharp(A) = p$.

Suppose at least two rows and two columns of A contains zero entries. Since $n \geqslant 4$, there are cells $E_1, E_2 \not\sqsubseteq A$ and $E_3 \sqsubseteq A$ such that $\mathrm{fr}(E_1 + E_2 + E_3) = 3$. If $\sharp(T^*(A)) = \sharp(T^*(A \setminus E_3))$, then $T^*(A) = T^*(A \setminus E_3)$ and so

$$\begin{aligned} T^*(J \setminus (E_1 + E_2)) &= T^*(A) + T^*(J \setminus (A + E_1 + E_2)) \\ &= T^*(A \setminus E_3) + T^*(J \setminus (A + E_1 + E_2)) \\ &= T^*(J \setminus (E_1 + E_2 + E_3)), \end{aligned}$$

a contradiction by Proposition 3.4. Then by assumption, we have $\sharp(T^*(A)) > \sharp(T^*(A \setminus E_3)) \geqslant \sharp(A \setminus E_3) = p - 1$. Thus, the result follows.

Now suppose A has zero entries in one row only. Since $\sharp(A) \leqslant mn - 2$, without loss of generality we may assume that A has zero entries in the first row with zero $(1, 1)$th and $(1, 2)$th entries. By assumption, $\sharp(T^*(A)) \geq \sharp(T^*(A \setminus E_{21})) \geqslant \sharp(A \setminus E_{21}) = p - 1$. Suppose $\sharp(T^*(A)) = p - 1$. Take

$$G_1 = E_{21} + E_{33}, \quad G_2 = E_{21} + E_{34}, \quad G_3 = E_{22} + E_{33} \quad \text{and} \quad G_4 = E_{22} + E_{34}.$$

We claim that there is an index i in $\{1, 2, 3, 4\}$ such that

$$T^*(A \setminus G_i) = T^*(A).$$

If the claim holds, we take $F = \begin{cases} E_{12} & \text{if } i \in \{1, 2\}, \\ E_{11} & \text{if } i \in \{3, 4\}. \end{cases}$ Then $\mathrm{fr}(F + G_i) = 3$ and

$$\begin{aligned} T^*(J \setminus F) &= T^*(A) + T^*(J \setminus (A + F)) = T^*(A \setminus G_i) + T^*(J \setminus (A + F)) \\ &= T^*(J \setminus (F + G_i)), \end{aligned}$$

a contradiction by Proposition 3.4. Thus, $\sharp(T^*(A)) \geqslant p$ and the result follows.

It remains to prove our claim. If the claim does not hold, then for each $i \in \{1, 2, 3, 4\}$,

$$p - 1 = \sharp(T^*(A)) > \sharp(T^*(A \setminus G_i)) \geqslant \sharp(A \setminus G_i) = p - 2.$$

That is, $\sharp(T^*(A \setminus G_i)) = p - 2$ and so $T^*(A \setminus G_i) = T^*(A) \setminus H_i$ for some cell $H_i \sqsubseteq T^*(A)$. Notice that for any $i \neq j$, $(A \setminus G_i) + (A \setminus G_j)$ equals either A or $A \setminus E$ for some $E \in \{E_{21}, E_{22}, E_{33}, E_{34}\}$. Since $p - 1 = \sharp(T^*(A)) \geqslant \sharp(T^*(A \setminus E)) \geqslant p - 1$, we have $T^*(A \setminus E) = T^*(A)$ and so $T^*((A \setminus G_i) + (A \setminus G_j)) = T^*(A)$ in both cases. Then

$$\begin{aligned} T^*(A) &= T^*((A \setminus G_i) + (A \setminus G_j)) = T^*(A \setminus G_i) + T^*(A \setminus G_j) \\ &= (T^*(A) \setminus H_i) + (T^*(A) \setminus H_j). \end{aligned}$$

Thus, we must have $H_i \neq H_j$. Hence $\sharp(H_1 + H_2 + H_3 + H_4) = 4$.

Now let $B = A \setminus (E_{21} + E_{22} + E_{33} + E_{34})$. Since $T^*(B) \sqsubseteq T^*(A \setminus G_i)$ and $H_i \not\sqsubseteq T^*(A \setminus G_i)$, we have $H_i \not\sqsubseteq T^*(B)$ for all $i = 1, \ldots, 4$ and hence $H_1 + H_2 + H_3 + H_4 \not\sqsubseteq T^*(B)$. Then $T^*(B) \sqsubseteq T^*(A) \setminus (H_1 + H_2 + H_3 + H_4)$ and hence by assumption,

$$\begin{aligned} p - 4 &= \sharp(B) \leqslant \sharp(T^*(B)) \leqslant \sharp(T^*(A) \setminus (H_1 + H_2 + H_3 + H_4)) \\ &= (p - 1) - 4 = p - 5 \end{aligned}$$

which is impossible. Therefore, our claim holds.

Finally suppose A has zero entries in one column only. Without loss of generality we may assume that A has zero entries in the first column with zero (1, 1)th and (2, 1)th entries. Then the result follows by a similar argument with $G_1 = E_{12} + E_{33}$, $G_2 = E_{12} + E_{34}$, $G_3 = E_{22} + E_{33}$ and $G_4 = E_{22} + E_{34}$. □

Corollary 3.10. *The map $T^*|_{\mathscr{N}}$ is bijective from $\mathscr{N}$ onto $\mathscr{N}$.*

Proof. Suppose $T^*(N) = T^*(M)$ for some distinct $N, M \in \mathscr{N}$. Then

$$T^*(N + M) = T^*(N) + T^*(M) = T^*(N) \notin \mathscr{R}(\mathbb{S})^*.$$

But $\sharp(N + M) > \sharp(N) = k_{\max}$ which contradicts the definition of $k_{\max}$. Thus, T^* is injective in $\mathscr{N}$. Also by Lemmas 3.6–3.9, $T^*(\mathscr{N}) \subseteq \mathscr{N}$. Since $\mathscr{N}$ is finite, $T^*(\mathscr{N}) = \mathscr{N}$. Thus, the result follows. □

Lemma 3.11. *For any cell E, $T^*(E)$ is a cell. Furthermore, T^* is bijective on the set of cells.*

Proof. Let E be an arbitrary cell. Notice that for any $N \in \mathscr{N}$, as $T^*(\mathscr{N}) = \mathscr{N}$,

$$E \sqsubseteq N \in \mathscr{N} \;\Leftrightarrow\; E + N \in \mathscr{N} \;\Leftrightarrow\; T^*(E) + T^*(N) \in \mathscr{N} \;\Leftrightarrow\; T^*(E) \sqsubseteq T^*(N) \in \mathscr{N}.$$

Hence $T^*(\{N \in \mathscr{N} : E \sqsubseteq N\}) = \{N \in \mathscr{N} : T^*(E) \sqsubseteq N\}$. Since T^* is bijective on $\mathscr{N}$, the two sets $\{N \in \mathscr{N} : E \sqsubseteq N\}$ and $\{N \in \mathscr{N} : T^*(E) \sqsubseteq N\}$ have the same number of elements. This is possible only if $T^*(E)$ is a cell. Thus, the first assertion follows. The last assertion follows by Proposition 3.5. □

A matrix L is called a *line matrix* if $L = \sum_{k=1}^{n} E_{i,k}$ or $\sum_{l=1}^{m} E_{l,j}$ for some $i \in \{1, \ldots, m\}$ or $j \in \{1, \ldots, n\}$; $R_i = \sum_{k=1}^{n} E_{i,k}$ is an ith *row matrix* and $C_j = \sum_{l=1}^{m} E_{l,j}$ is a jth *column matrix*.

Corollary 3.12. *T^* preserves all line matrices.*

Proof. By Lemma 3.11, T^* is bijective on the set of cells. If T^* does not map some line matrix into a line matrix, without loss of generality, we assume that $T^*(E_{1,1}) = E_{1,1}$ and $T^*(E_{1,2}) = E_{2,2}$.

Case 1. $E_{1,1} + E_{1,2} + E_{2,2} \notin \mathscr{R}(\mathbb{S})^*$: Consider a matrix $X = E_{1,1} + E_{1,2} + E_{i,j}$, where $i \geqslant 2$ and $j \leqslant 2$. Then $X \notin \mathscr{R}(\mathbb{S})^*$. Since T^* strongly preserves $\mathscr{R}(\mathbb{S})^*$, $T^*(X) \notin \mathscr{R}(\mathbb{S})^*$ and hence $T^*(E_{i,j}) = E_{1,2}$ or $E_{2,1}$ for all $i \geqslant 2$ and $j \leqslant 2$. This contradicts Lemma 3.11.

Case 2. $E_{1,1} + E_{1,2} + E_{2,2} \in \mathscr{R}(\mathbb{S})^*$: Consider a matrix $X = E_{1,1} + E_{1,2} + E_{1,3}$. Then we have $\mathrm{fr}(T(X)) = 2$ or 3. By Proposition 2.5, there is a matrix B with $\sharp(B) = 2$ such that $(T(X) + B)^* \notin \mathscr{R}(\mathbb{S})^*$. Furthermore we can write $B = T(C)$ for some matrix C with $\sharp(C) = 2$ so that $T(X) + B = T(X + C)$. But then $(X + C)^* \in \mathscr{R}(\mathbb{S})^*$ by Proposition 2.4, contradicting that T^* strongly preserves $\mathscr{R}(\mathbb{S})^*$.

Therefore T^* preserves all line matrices. $\square$

An operator T on $\mathscr{M}_{m,n}(\mathbb{S})$ is called a (P, Q, B)-*operator* if there are permutation matrices P and Q, and a matrix B with $B^* = J$ such that $T(X) = P(X \circ B)Q$ for all $X \in \mathscr{M}_{m,n}(\mathbb{S})$, or $m = n$ and $T(X) = P(X^{\mathrm{t}} \circ B)Q$ for all $X \in \mathscr{M}_n(\mathbb{S})$.

Now, we are ready to prove the main theorem.

Theorem 3.13. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{S})$ with $\min\{m, n\} \geqslant 3$. If T strongly preserves regularity, then T is a (P, Q, B)-operator.*

Proof. Suppose that T strongly preserves regularity. Then T^* is bijective on the set of cells by Lemma 3.11 and T^* preserves all line matrices by Corollary 3.12. Since no combination of s row matrices and t column matrices can dominate $J_{m,n}$ where $s + t = \min\{m, n\}$ unless $s = 0$ or $t = 0$, we have that either

(1) the image of T^* of each row matrix is a row matrix and the image of T^* of each column matrix is a column matrix, or
(2) the image of T^* of each row matrix is a column matrix and the image of T^* of each column matrix is a row matrix.

If (1) holds, then there are permutations σ and τ of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively such that $T^*(R_i) = R_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let P and Q be permutation matrices corresponding to σ and τ, respectively. Then we have

$$T(E_{i,j}) = b_{i,j} E_{\sigma(i),\tau(j)} = P(b_{i,j} E_{i,j})Q,$$

where $b_{i,j} \neq 0$ for all cells $E_{i,j}$. By the action of T on the cells, we have $T(X) = P(X \circ B)Q$.

If (2) holds, then $m = n$ and a parallel argument shows that there are permutation matrices P and Q, and a matrix B with $B^* = J$ such that $T(X) = P(X^{\mathrm{t}} \circ B)Q$ for all $X \in \mathscr{M}_n(\mathbb{S})$. $\square$

Corollary 3.14. *Let T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{B})$ with $\min\{m, n\} \geqslant 3$. Then T strongly preserves regularity if and only if there are permutation matrices P and Q such that $T(X) = PXQ$ for all $X \in \mathscr{M}_{m,n}(\mathbb{B})$, or $m = n$ and $T(X) = PX^{\mathrm{t}}Q$ for all $X \in \mathscr{M}_n(\mathbb{B})$.*

Proof. It follows from Theorem 3.13 and Proposition 2.1 $\square$

4. Other results

In this section, we have characterizations of linear operators that strongly preserve regularity over several semirings such as the nonnegative integers $\mathbb{Z}_+$, the nonnegative reals $\mathbb{R}_+$ and chain semiring $\mathbb{C}$ including the fuzzy scalars.

A matrix A in $\mathscr{M}_n(\mathbb{S})$ is called *monomial* if A^* is a permutation matrix. It is well known that a monomial matrix A is invertible if and only if all nonzero elements of A are units in $\mathbb{S}$.

Let $\mathbb{P}_+$ be the nonnegative part of a subring $\mathbb{P}$ with identity of the reals. The nonnegative integers $\mathbb{Z}_+$, and the nonnegative reals $\mathbb{R}_+$ are good examples of $\mathbb{P}_+$.

Proposition 4.1. *Let a, b, c and d be units in $\mathbb{P}_+$. Then $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is regular over $\mathbb{P}_+$ if and only if $ad = bc$.*

Proof. If $ad = bc$, then we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Conversely assume that X is regular. Then there is a nonzero matrix $Y = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, say $x \neq 0$ such that $XYX = X$: that is,

$$\begin{bmatrix} a(ax+by)+b(az+cw) & a(bx+dy)+b(bz+dw) \\ c(ax+by)+d(az+cw) & c(bx+dy)+d(bz+dw) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

From $(1,2)$th and $(2,2)$th entries of XYX and X, we have $ab^{-1}(bx+dy)+(bz+dw) = 1 = cd^{-1}(bx+dy)+(bz+dw)$, and hence $ab^{-1}(bx+dy) = cd^{-1}(bx+dy)$. Since $bx+dy \neq 0$, it follows by the cancellation property that $ab^{-1} = cd^{-1}$, equivalently $ad = bc$. $\square$

Theorem 4.2. *Let $\min\{m,n\} \geqslant 3$ and T be a linear operator on $\mathscr{M}_{m,n}(\mathbb{P}_+)$. Then T strongly preserves regularity if and only if there are invertible matrices U and V such that $T(X) = UXV$ for all $X \in \mathscr{M}_{m,n}(\mathbb{P}_+)$, or $m = n$ and $T(X) = UX^{\mathrm{t}}V$ for all $X \in \mathscr{M}_n(\mathbb{P}_+)$.*

Proof. By Proposition 2.1, the sufficiency is obvious. To prove the necessity, assume that T strongly preserves regularity. By Theorem 3.13, there are permutation matrices P and Q, a matrix B with $B^* = J$ such that $T(X) = P(X \circ B)Q$ for all $X \in \mathscr{M}_{m,n}(\mathbb{P}_+)$, or $m = n$ and $T(X) = P(X^{\mathrm{t}} \circ B)Q$ for all $X \in \mathscr{M}_n(\mathbb{P}_+)$. For the case of $T(X) = P(X \circ B)Q$, we define the operator L on $\mathscr{M}_{m,n}(\mathbb{P}_+)$ by $L(X) = P^{\mathrm{t}}T(X)Q^{\mathrm{t}} = X \circ B$. Since T strongly preserves regularity, so does L. By Lemma 2.8, all entries of B are regular and hence units because only units are nonzero regular elements over $\mathbb{P}_+$.

If $\mathrm{fr}(B) \neq 1$, there is a 2×2 submatrix C of B such that $\mathrm{fr}(C) = 2$. Without loss of generality, we assume that $C = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$. Then $b_{1,1}b_{2,2} \neq b_{1,2}b_{2,1}$ and hence C is not regular by Proposition 4.1. Consider a matrix $Y = E_{1,1} + E_{1,2} + E_{2,1} + E_{2,2}$. Then clearly Y is regular, while $L(Y) = \begin{bmatrix} C & O \\ O & O \end{bmatrix}$ is not regular by (2.2), a contradiction. Hence $\mathrm{fr}(B) = 1$. By Lemma 2.8, there are diagonal matrices D and E such that $L(X) = DXE$ for all $X \in \mathscr{M}_{m,n}(\mathbb{P}_+)$. Since all entries of B are units, all diagonal entries of D and E are units. Since $L(X) = P^{\mathrm{t}}T(X)Q^{\mathrm{t}} = X \circ B$, we have $T(X) = PDXEQ$. If we let $U = PD$ and $V = EQ$, then $U \in \mathscr{M}_m(\mathbb{P}_+)$ and $V \in \mathscr{M}_n(\mathbb{P}_+)$ are invertible. Thus we have $T(X) = UXV$ for all $X \in \mathscr{M}_{m,n}(\mathbb{P}_+)$.

If $m = n$ and T is of the form $T(X) = P(X^{\mathrm{t}} \circ B)Q$, then a parallel argument shows that there are invertible matrices U and V such that $T(X) = UX^{\mathrm{t}}V$ for all $X \in \mathscr{M}_n(\mathbb{P}_+)$. $\square$

Corollary 4.3. *Let* $\min\{m, n\} \geqslant 3$ *and* T *be a linear operator on* $\mathscr{M}_{m,n}(\mathbb{Z}_+)$. *Then* T *strongly preserves regularity if and only if there are permutation matrices* P *and* Q *such that* $T(X) = PXQ$ *for all* $X \in \mathscr{M}_{m,n}(\mathbb{Z}_+)$, *or* $m = n$ *and* $T(X) = PX^{\mathrm{t}}Q$ *for all* $X \in \mathscr{M}_n(\mathbb{Z}_+)$.

Let $\mathbb{C}$ be any chain semiring.

Proposition 4.4. *Let* $A = \begin{bmatrix} p & q \\ q & 0 \end{bmatrix}$ *be a matrix in* $\mathscr{M}_2(\mathbb{C})$ *with* $pq \neq 0$. *Then* A *is regular if and only if* $pq = p$.

Proof. If $pq = p$, then $\begin{bmatrix} p & q \\ q & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ q & 0 \end{bmatrix} = \begin{bmatrix} p & q \\ q & 0 \end{bmatrix}$ and hence A is regular.

Conversely, assume that A is regular and $pq \neq p$. Then $p \neq q$, $pq = q$ (i.e., $q < p$) and there is a nonzero $G = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathscr{M}_2(\mathbb{C})$ such that $AGA = A$ and hence

$$AGA = \begin{bmatrix} px + q(y + z + w) & q(x + z) \\ q(x + y) & qx \end{bmatrix} = \begin{bmatrix} p & q \\ q & 0 \end{bmatrix} = A.$$

From $(2, 2)$th entries of AGA and A, $x = 0$ since $q \neq 0$. Again from $(1, 1)$th entries of AGA and A, $q(y + z + w) = p$. But this is impossible because $q < p$. Therefore A is not regular for $pq \neq p$. $\square$

Note that if A is a monomial matrix in $\mathscr{M}_n(\mathbb{C})$, then A is invertible if and only if A is a permutation matrix because 1 is the only unit element in $\mathbb{C}$.

Theorem 4.5. *Let* $\min\{m, n\} \geqslant 3$ *and* T *be a linear operator on* $\mathscr{M}_{m,n}(\mathbb{C})$. *Then* T *strongly preserves regularity if and only if there are permutation matrices* P *and* Q *such that* $T(X) = PXQ$ *for all* $X \in \mathscr{M}_{m,n}(\mathbb{C})$, *or* $m = n$ *and* $T(X) = PX^{\mathrm{t}}Q$ *for all* $X \in \mathscr{M}_n(\mathbb{C})$.

Proof. The sufficiency follows Proposition 2.1. For the necessary, assume that T strongly preserves regularity. By Theorem 3.13, there are permutation matrices P, Q and a matrix B with $B^* = J$ such that $T(X) = P(X \circ B)Q$ for all $X \in \mathscr{M}_{m,n}(\mathbb{C})$, or $m = n$ and $T(X) = P(X^{\mathrm{t}} \circ B)Q$ for all $X \in \mathscr{M}_n(\mathbb{C})$.

Let T be of the form $T(X) = P(X \circ B)Q$. Without loss of generality, we assume that $P = I_m$ and $Q = I_n$ so that $T(X) = X \circ B$ and $T(J) = B$. Now we will show that $B = J$, equivalently $b_{i,j} = 1$ for all i and j. It is sufficient to consider $b_{1,1}$; for $b_{i,j}$ is any entry of $T(J)$, let P' be the transposition matrix that exchanges 1st and ith rows from identity matrix I_m, and Q' the transposition matrix that exchanges 1st and jth rows from identity matrix I_n. Define a linear operator L on $\mathscr{M}_{m,n}(\mathbb{C})$ by $L(X) = P'T(X)Q'$ for all X. Since T strongly preserves regularity, so does L. Furthermore the $(1, 1)$th entry of $L(J)$ is $b_{i,j}$.

If $b_{1,1} \neq 1$, let $\alpha = \min\{b_{1,1}, b_{1,2}, b_{2,1}\}$. Then $\alpha \neq 0, 1$. Consider a matrix $A = E_{1,1} + \alpha(E_{1,2} + E_{2,1})$. By Proposition 4.4, A is not regular and hence $T(A) = b_{1,1}E_{1,1} + \alpha(E_{1,2} + E_{2,1})$ is not regular so that $b_{1,1}\alpha = \alpha$ and $b_{1,1} \neq \alpha$. Thus $\alpha = b_{1,2}$ or $b_{2,1}$. If $\alpha = b_{1,2}$, consider a matrix $A_1 = b_{1,1}(E_{1,1} + E_{1,2}) + \alpha E_{2,1}$. Then A_1 is regular because $A_1[b_{1,1}(G_{1,2} + G_{2,1})]A_1 = A_1$, where $G_{i,j}$ are cells in $\mathscr{M}_{n,m}(\mathbb{C})$. But $T(A_1) = b_{1,1}E_{1,1} + \alpha(E_{1,2} + E_{2,1})$ is not regular by Proposition 4.4, a contradiction. For the case $\alpha = b_{2,1}$, if we consider a matrix $A_2 = b_{1,1}(E_{1,1} + E_{2,1}) + \alpha E_{1,2}$, then A_2 is regular while $T(A_2)$ is not regular, a contradiction. Therefore $b_{1,1} = 1$. Hence $B = J$. Therefore $T(X) = PXQ$ for all $X \in \mathscr{M}_{m,n}(\mathbb{C})$.

For the case of $m = n$ and $T(X) = P(X^{\mathrm{t}} \circ B)Q$, a parallel argument shows that $B = J$ so that $T(X) = PX^{\mathrm{t}}Q$ for all $X \in \mathscr{M}_n(\mathbb{C})$. $\square$

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